CORRESPONDENCE

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A Note on Forced Equatorial Waves [1]

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1. INTRODUCTION

Recent spectral analyses of data from stations in the tropical Pacific (Wallace and Chang 1969 and Chang et al. 1970) have confirmed the existence of westward-propagating disturbances in the lower and middle troposphere with wavelengths of 3000 to 5000 km and periods relative to the ground of 4 to 5 days. The observed vertical structure of these easterly waves depends on the season and longitude. In general, however, the waves have strong tilt with height in the central Pacific corresponding to vertical wavelengths of about 10 km. As the waves propagate westward, the vertical wavelengths appear to increase, so that the waves become almost barotropic in structure in the western Pacific. The purpose of this note is to show that these waves can be interpreted theoretically as *forced equatorial Rossby waves*.

2. FORCED EQUATORIAL WAVES

Lindzen (1967) has shown that a combination of two β-planes, one centered at the Equator and the other centered at a middle latitude, can be used to formulate an approximate theory for both forced and free atmospheric oscillations with periods in excess of one-half day. In particular, he showed that the equatorial β-plane can satisfactorily approximate solutions on the sphere for modes that decay sufficiently rapidly away from the Equator. In the case of free oscillations, only the lowest latitudinal modes yield valid approximations on the equatorial β-plane. The situation for forced waves is more complicated. The equivalent depth, h, corresponding to a forced mode with longitudinal wave number, s, and Doppler-shifted frequency, ω, on the equatorial β-plane is given by equation (36) of Lindzen (1967) as

$$\sqrt{gh}=\frac{\Omega a(2n+1)}{s^2\left(\frac{2\Omega}{s\omega}-1\right)}\left[1\pm\left(1-\left(\frac{\omega}{\Omega}\right)^2\left(\frac{s}{2n+1}\right)^2\left(\frac{2\Omega}{s\omega}-1\right)\right)^{1/2}\right] \quad (1)$$

where Ω is the angular velocity of rotation of the earth, g is the gravitational acceleration, a is the radius of the earth, and n is the meridional mode number. As shown in Lindzen (1967), the equatorial β-plane solution corresponding to a given h will begin to decay at a distance, y_a, from the Equator where

$$y_a^2=\frac{(2n+1)\sqrt{gh}}{(2\Omega/a)}. \quad (2)$$

Hence, solutions corresponding to the negative root in equation (1) decay more rapidly away from the Equator than do solutions corresponding to the positive root. Lindzen (1967) concludes as follows: "In practice, it has usually happened that (the solution) is not a valid approximation when the plus sign obtains." The *forced* equatorial waves discussed in Lindzen (1967) are limited to the modes that have equivalent depths given by the negative root in equation (1).

The distinction between the modes given by the two roots of equation (1) has been further elucidated by Lindzen and Matsuno (1968). They point out that the positive root corresponds to an internal Rossby wave, while the negative root corresponds to an internal gravity wave. (The $n=0$ mode turns out to require special treatment.) The following properties of the solution corresponding to the positive root are significant for the present discussion:

a) The h^+ is infinite for $\omega=2\Omega/s$.

b) For $\omega>2\Omega/s$, or $\omega<0$, the positive root yields $\sqrt{gh}<0$ that corresponds to a solution which is unbounded as $y\rightarrow\pm\infty$, and hence invalid.

c) The equivalent depth h^+ decreases rapidly for decreasing ω when $0<\omega<2\Omega/s$. Thus, we see by applying the condition given in (2) that, for $0<\omega<2\Omega/s$ and for sufficiently small n, the positive root of equation (1) gives valid solutions. This point is illustrated in figure 1 in which the equivalent depths, h^+, corresponding to the positive root in (1) are shown as a function of period and longitudinal wave number for the $n=1$ mode.

3. DISCUSSION

In applying equation (1) to compute equivalent depths for the observed waves, it is necessary to compute the Doppler shifted frequency of the observed waves, since ω in equation (1) is the frequency relative to the mean zonal flow. Unfortunately, the zonal wind has both horizontal and vertical shear so that it is difficult to estimate the effective Doppler shifted frequency.

According to Reed (1970), the mean zonal flow at 700 mb is typically easterly at about 8 m sec^{-1} in the

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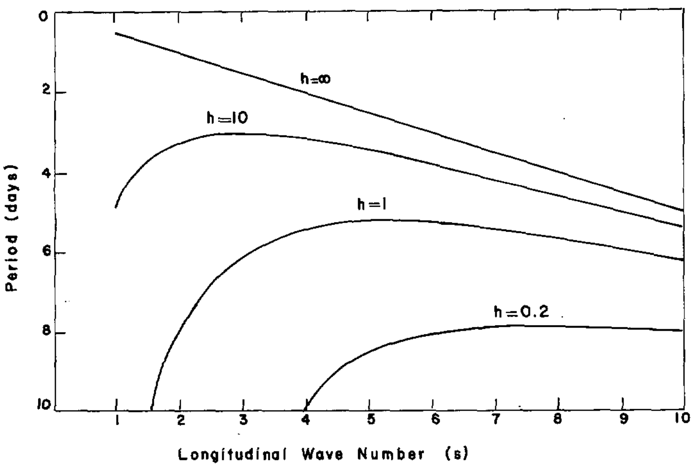

FIGURE 1.—The equivalent depth h^+ in kilometers as a function of period and longitudinal wave number for the $n=1$ mode and westward-propagating waves.

central Pacific, but close to zero in the western Pacific (although it is still easterly below 700 mb in the western Pacific). Thus, the period of the waves relative to the zonal flow in the lower troposphere is quite long in the central Pacific (probably in the range of 8 to 12 days) but rather shorter in the western Pacific (in the range of 6 to 8 days).

Since the observations reported in Wallace and Chang (1969) and Chang et al. (1970) are mainly confined to a rather narrow latitude band in the Northern Hemisphere, it is not possible to deduce the latitudinal mode number of the waves directly from the data. However, for periods in the range of 8–12 days and a wavelength of 4000 km (longitudinal wave number 10), the $n=0$ mode solutions have vertical wavelengths much shorter than those observed. Therefore, it seems likely that the observed waves correspond to the $n=1$ mode oscillation.

In table 1, the equivalent depths corresponding to both the positive (h^+) and negative (h^-) roots in equation (1) are shown for $n=1$, $s=10$, and periods of 7 and 10 days. Also indicated in the table is the latitude, y_d, at which the wave begins to decay, and the vertical wavelength corresponding to the calculated equivalent depth. The vertical wavelength, L, is computed from the formula

$$L\approx 2\Omega\Big/\left[\frac{1}{hT_0}\left(\frac{g}{c_p}+\frac{d\overline{T}}{dz}\right)-1/4H^2\right]^{1/2} \tag{3}$$

where T_0 is a constant average temperature, $H=RT_0/g$ is the scale height, and $d\overline{T}/dz$ is an average lapse rate. Equation (3) is merely a modification for a thermally stratified atmosphere of the expression for vertical wavelength Lindzen (1967) given for the isothermal case. The vertical wavelengths shown in table 1 were obtained by setting $T_0=300°K$ and $d\overline{T}/dz=-6°C/km$ in equation (3).

TABLE 1.—*Equivalent depths for the h^+ and h^- roots in equation (1) for periods of 7 and 10 days and $n=1$, $s=10$*

Period	7 days		10 days	
	h^+	h^-	h^+	h^-
Equivalent depth (km)	0.47	2.6×10^{-4}	0.075	6.7×10^{-5}
Vertical wavelength (km)	39	0.88	15	0.44
y_d (km)	3,000	460	1,900	320

The theoretically deduced vertical and lateral scales shown in table 1 are in order of magnitude agreement with those of the observed easterly waves provided that the h^+ root is chosen. In particular, the increase in vertical wavelength as the Doppler shifted frequency increases is consistent with the observed change in vertical structure as the waves propagate into the western Pacific. It thus appears that the observed easterly waves are in fact forced equatorial Rossby waves. The forcing for the observed waves is probably provided by the release of latent heat in the active precipitation areas associated with the observed waves as suggested in Wallace and Chang (1969) and elsewhere. However, it remains to determine why the observed frequency wave-number combination is excited in the atmosphere, while apparently very little energy goes into forced Rossby waves of other periods and wavelengths.

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